Exam Algebraic Structures
June 21st, 2018, 14:00 – 17:00,
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This exam consists of three exercises, and each of them has six subproblems. You get 6 points for free, and you can score 3 points for every correctly answered subproblem. In this way you can obtain at most 60 point for the total exam. Only answers given with a complete argument will be considered!

(1) Consider the subring $\mathbb{Z}[\sqrt{3}] := \{a + b\sqrt{3} \ : \ a, b \in \mathbb{Z}\} \subset \mathbb{C}$.
  (a) Find an element $\neq \pm 1$ in $\mathbb{Z}[\sqrt{3}]^\times$.
  (b) Show that $\mathbb{Z}[\sqrt{3}]^\times$ is an infinite set.
  (c) Prove that the ring $\mathbb{Z}[\sqrt{3}]$ is Euclidean, with as function $g\colon \mathbb{Z}[\sqrt{3}] \to \mathbb{Z}_{\geq 0}$ the one defined as $g(a+b\sqrt{3}) := |a^2 - 3b^2|$.
  (d) Show that $\varphi\colon \mathbb{Z}[\sqrt{3}] \to \mathbb{F}_{11}$ given by the rule $\varphi(a+b\sqrt{3}) = (a + 5b) \bmod 11$, is a homomorphism of rings.
  (e) With φ as in (d), show that $\mathrm{Ker}(\varphi) = \mathbb{Z}[\sqrt{3}] \cdot 11 + \mathbb{Z}[\sqrt{3}] \cdot (-5 + \sqrt{3})$.
  (f) Again with notations as in (d), find (prove your claim!) $\alpha \in \mathbb{Z}[\sqrt{3}]$ such that $\mathrm{Ker}(\varphi) = \mathbb{Z}[\sqrt{3}] \cdot \alpha$.

(2) By $\mathrm{M}_2(\mathbb{F}_2)$ we denote the ring consisting of all 2×2 matrices with coefficients in $\mathbb{F}_2$. Let $\varphi = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{F}_2)$. Furthermore, $\mathrm{ev}_\varphi\colon \mathbb{F}_2[x] \to \mathrm{M}_2(\mathbb{F}_2)$ denotes evaluation at φ.
  (a) Is ev_φ surjective?
  (b) Show that $x^2 + x + 1$ is the minimal polynomial of φ.
  (c) Is $\mathrm{Ker}(\mathrm{ev}_\varphi)$ a prime ideal?
  (d) Is $\mathrm{ev}_\varphi(\mathbb{F}_2[x])$ a field?
  (e) Determine a generator of the ideal $\mathrm{Ker}(\mathrm{ev}_\varphi) \subset \mathbb{F}_2[x]$.
  (f) How many elements does $(\mathbb{F}_2[x]/\mathrm{Ker}(\mathrm{ev}_\varphi))^\times$ have?

(3) Consider $f := x^3 - 2 \in \mathbb{F}_7[x]$ and $R := \mathbb{F}_7[x]/f \cdot \mathbb{F}_7[x]$.
  (a) Show that $f \in \mathbb{F}_7[x]$ is irreducible.
  (b) Prove that R is a field.
  (c) Prove that $R \cong \mathbb{F}_7[y]/(y^3 - y + 2)$.
  (d) How many elements does the splitting field of f over $\mathbb{F}_7$ have?
  (e) Let α be a zero of f in the splitting field Ω of f over $\mathbb{F}_7$. What is the order of $\alpha \in \Omega^\times$?
  (f) By $\varphi(a + bx + cx^2 \bmod f) = a + 4bx + 2cx^2 \bmod f$ we define a map $\varphi\colon R \to R$. Show that φ is an automorphism of the ring R, and that φ has order 3 in $\mathrm{Aut}(R)$.